Crossover from Selberg's type to Ruelle's type zeta function in classical kinetics

Daniel L. Miller*
Department of Physics of Complex Systems, The Weizmann Institute of Science, Rehovot 76100, Israel
(Received 2 March 1998)

The decay rates of the density-density correlation function are computed for a chaotic billiard with some disorder inside. In the case of the clean system the rates are zeros of Ruelle's zeta function and in the limit of strong disorder they are roots of Selberg's zeta function. We constructed the interpolation formula between two limiting zeta functions by analogy with the case of the integrable billiards. The almost clean limit is discussed in some detail. [S1063-651X(99)03103-7]

PACS number(s): 05.45.−a, 05.20.Dd, 51.10.+y

It is natural to assume that chaotic billiards with a small amount of disorder [1–3] are good models for ballistic cavities, which have been employed in a number of recent experiments, see Ref. [4]. Such a model is interesting because the disorder (in two dimensions) can be characterized by one parameter: the elastic scattering time τ. The mixing properties of this model in two important limits $\tau\to 0$ and $\tau\to\infty$ were known in the literature, see Refs. [5,6]. In the present paper we discuss the crossover from one limit to the other for some two-dimensional billiard. The case of the three-dimensional billiards is more complicated. Particularly, the uniform scattering in three dimensions leads to very fast resonant mixing. At the end of the paper we provide the generalization of our results for the case of three dimensions.

Let us focus attention on the eigenvalues and the eigenmodes of the kinetic equation for the distribution function $f(\vec{r},\phi)$ of noninteracting particles inside a two-dimensional billiard. This function is defined on the constant energy manifold $|\vec{v}_\phi|=v=\text{const}$, and $\vec{v}_\phi=(v\cos(\phi),v\sin(\phi))$. The precise form of the kinetic equation depends on the details of the impurity potential, but we are going to investigate two models,

$$\frac{\partial f}{\partial t}+\vec{v}_\phi\cdot\vec{\nabla}f=\begin{cases}\dfrac{\overline{f}-f}{\tau}, & \text{model 1}\\[2ex] \dfrac{1}{\tau}\dfrac{\partial^2 f}{\partial\phi^2}, & \text{model 2},\end{cases} \tag{1}$$

where $\overline{f}(\vec{r})=\int f(\vec{r},\phi)d\phi/(2\pi)$. The above equation has to be solved with mirror boundary conditions $f(\vec{r},\phi)=f(\vec{r},2\alpha(\vec{r})-\pi-\phi)$, where $\vec{r}$ is taken on the boundary of the billiard and $\vec{n}=(\cos(\alpha),\sin(\alpha))$ is normal to the boundary. Equation (1) has a special solution $f_0(\vec{r},\phi)=\text{const}$ for all values of τ and we will ignore it in the rest of the paper.

In both models, the collision integral conserves energy. The first model corresponds to uniform scattering in all directions and the second model is valid if small angle scattering is dominant. There is a relatively simple analytical treatment of these two limiting cases. In the same time it will give us the qualitative understanding of the general case.

Let us look for solutions proportional to $e^{-s_n t}$. The eigenvalues s_n of the kinetic equation are so-called mixing rates, or decay rates of the density-density correlation function when $\tau\to 0$ or Ruelle's resonances when $\tau\to\infty$. These resonances can be found as zeros of the spectral determinant $Z(s)$. Let us start to compute $Z(s)$ in the limit of pure chaos. Cvitanovic and Eckardt [7] have computed the spectral determinant for the axiom A system, but the result is the expansion over the unstable periodic orbits and it seems to be valid for a wide class of systems. Therefore, in the limit $\tau\to\infty$, the spectral determinant is

$$-\ln(Z(s))=\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{1}{|\det(I-M_p^r)|}e^{sl_p r/v}, \tag{2}$$

where v is the velocity. This expression contains the sum over the primitive periodic orbits p taken with repetition r. In the case of billiards, the action for the periodic trajectory is just the length $l_p r$. Each oscillating term in the sum in Eq. (2) is weighted by the stability amplitude, which behaves on average, as

$$\frac{1}{|\det(I-M_p^r)|}\equiv e^{-\lambda_{pr}l_p r/v}\approx e^{-\lambda l_p r/v}, \tag{3}$$

where the first equality defines λ_{pr}, and λ is the Lyapunov exponent of the billiard. The zeta function given by Eq. (2) is of Ruelle's type.

When the disorder is strong, the kinetic equation can be transformed into the diffusion equation for $\overline{f}$,

$$\frac{\partial\overline{f}}{\partial t}-\frac{v^2\tau}{2}\nabla^2\overline{f}=0, \tag{4}$$

which has to be solved with boundary conditions $\vec{n}\cdot\vec{\nabla}\overline{f}=0$. This approximation is valid if the spatial variation of the initial distribution is small on the scale of the mean-free-path τv. Equation (4) allows one to find the decay of modes with $\overline{f}\not\equiv 0$. The decay rates for modes with $\overline{f}=0$ for all $\vec{r}$ should be computed in a different way.

In the limit $\tau\to 0$ we can use the "semiclassical" approximation for equation

$$(k^2+\nabla^2)\overline{f}=0, \tag{5}$$

*Electronic address: fndaniil@wicc.weizmann.ac.il

where $k^2=2s/(v^2\tau)$. The logarithm of the Selberg's type zeta function is again the sum over periodic orbits [8,9],

$$-\ln(Z(s))=\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{1}{\sqrt{|\det(I-M_p^r)|}}e^{ikl_pr}. \qquad (6)$$

Here the phase factors correspond to the case when Eq. (5) should be solved with the Neumann boundary condition and Maslov's indexes vanish [10]. The natural question to ask is whether it is possible to compute the decay rates of the modes with $\overline{f}\not\equiv 0$ for all values of τ by constructing a suitable zeta function?

We can understand the connection between different types of zeta functions by making use of the following approximation:

$$-\ln(Z_n(x))=\sum_{pr}\frac{1}{r}e^{(x-\lambda_{pr}/n)l_pr/v}, \qquad (7)$$

$$Z_2(x)\approx Z_1(x+\lambda/2), \qquad (8)$$

where Z_1 is Ruelle's type zeta function, Z_2 is Selberg's type zeta function and Eq. (8) is known as a $\lambda/2$ shift. Such a shift was observed in the problem of quantum and classical scattering in a three-disk problem, compare Figs. 2.14 and 3.6 of Ref. [11] and see Refs. [12,13] for details. Equation (8) holds with high accuracy in various systems [14]; however, it is not yet known whether it is an approximation or an exact result.

Assuming that Eq. (8) is accurate enough, we have the relation for our zeta function being taken for different values of τ,

$$Z(s)=\begin{cases} Z_1(s)\approx Z_2\left(s-\dfrac{\lambda}{2}\right), & \tau\gg\dfrac{1}{|s|},\dfrac{1}{\lambda}\\[2ex] Z_2\left(i\sqrt{\dfrac{2s}{\tau}}\right), & \tau\ll\dfrac{1}{|s|},\dfrac{1}{\lambda}.\end{cases} \qquad (9)$$

We argue that the kinetic zeta function is the function Z_2 of the yet unknown combination of s, τ, and λ. This combination can be obtained for integrable billiards. The path integral method will help us to modify this combination in the chaotic case. As an immediate consequence of the relation Eq. (9), we have for the eigenvalues of the kinetic equation,

$$s_n=\begin{cases}\lambda/2+iq_nv, & \tau\to\infty\\ q_n^2v^2\tau/2, & \tau\to 0,\end{cases} \qquad (10)$$

where $q_n, n=1\ldots\infty$ are the eigenvalues of the wave number in Eq. (5). The full dependence s_n on q_n,τ and λ exists if our postulate is correct, and we will compute it.

Let's examine first the integrable case. Model 1 for the square billiard was solved by Atland and Gefen [1], and Agam and Fishman [2], who modeled the short-range random potential by random spheres or circles. For the square billiard of size L the spatial dependence of the density $\overline{f}(r)$ is $\Sigma\exp(i\vec{q}\cdot\vec{r})$, where q is such that $\sin(q_xL)\sin(q_yL)=0$, and the sum is over four possible directions of $\vec{q}$. Then the values of q are "quantized," and we will denote them q_n, and the modes with $\overline{f}\not\equiv 0$ can be numbered

$$f_n(\vec{r},\phi,t)\propto\sum\frac{\tau^{-1}}{-s+\tau^{-1}+i\vec{v}_\phi\vec{q}_n}e^{i\vec{q}_n\vec{r}-st}, \qquad (11)$$

where the sum is over four possible directions of $\vec{q}_n$. Integration over ϕ leads to the equation for s_n,

$$\tau^{-2}=(s_n-\tau^{-1})^2+v^2q_n^2, \qquad (12)$$

and the corresponding zeta function is

$$-\ln(Z(s))=\sum_p\sqrt{(L^4q/\pi^3l_p^3)}e^{iql_p}, \qquad (13)$$

where p is not a single orbit but the resonant tori [15], and the connection between s and q is as in Eq. (12).

In the case of model 2, the solutions are still proportional to $e^{i\vec{q}\cdot\vec{r}}$, but the angular dependence is different. The solution with $\overline{f}\not\equiv 0$ is the "ground state" of

$$\left[-s+i\vec{v}_\phi\cdot\vec{q}_n-\frac{1}{\tau}\frac{\partial^2}{\partial\phi^2}\right]f_n=0, \qquad (14)$$

because the real parts of the decay rates are positive. Surprisingly, Eq. (12) gives a numerically good approximation for s_n for this model. Other "angular" modes, which have $\overline{f}=0$, are very different for models 1 and 2.

It is not easy to compute eigenmodes of Eq. (1) for the integrable billiards, which have other than rectangular shapes. For such cases, Eq. (13) becomes an interpolation formula for the kinetic zeta function. One should only replace the pre-exponential factor by the amplitude from the Berry-Tabor [16] formula. For example, the resonant tori for the circular billiard of radius R are numbered by the winding number M and by the number of vertices n, have length $l_{Mn}=nL_{Mn}/\pi$, where $L_{Mn}=2\pi R\sin(\pi M/n)$. Then one can use Eq. (13) after the replacement $p\to Mn$, and $L\to L_{Mn}$, see Ref. [17].

Combining together Eqs. (2), (6), (12), and (13) we can introduce the kinetic zeta function as

$$-\ln(Z(s))=\sum_p\sum_{r=1}^{\infty}\frac{1}{r}e^{[\sqrt{(s-2/\tau_{pr})s}-\lambda_{pr}/2]\,l_pr/v}, \qquad (15a)$$

where

$$\tau_{pr}=\begin{cases}\tau, & \lambda_{pr}\tau/2<1\\ 2/\lambda_{pr}, & \lambda_{pr}\tau/2>1.\end{cases} \qquad (15b)$$

For $\lambda\tau<1/2$ the kinetic zeta function coincides with Selberg's type zeta function Eq. (6) in the domain of the complex s plane $|s\tau|\ll 1$. For $\lambda\tau>1/2$ the kinetic zeta function becomes independent of τ and coincides with Ruelle's type zeta function Eq. (2) in the domain of the complex s plane $|s\tau|\gg 1$.

The interpolation formula Eqs. (15) for the kinetic zeta function implies the following interpolation formula for the decay rates:

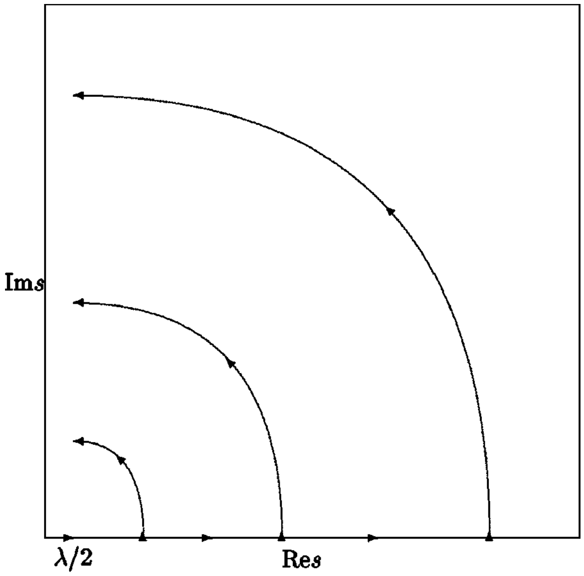

FIG. 1. The decay rates of the density-density correlation function move on the complex plane when disorder decreases, as shown by arrows. The termination point is $\mathrm{Re}s \approx \lambda/2$, where λ is the Lyapunov exponent.

$$s_n=\begin{cases} \frac{1}{\tau}-\sqrt{1/\tau^2-(q_n v)^2}, & \frac{1}{q_n v}\geqslant \tau \\ \frac{1}{\tau}+i\sqrt{(q_n v)^2-1/\tau^2}, & \frac{2}{\lambda}\geqslant \tau \geqslant \frac{1}{q_n v} \\ \frac{\lambda}{2}+iq_n v, & \tau \geqslant \frac{2}{\lambda}, \end{cases} \tag{16}$$

where q_n are the eigenvalues of the wave number in Eq. (5). There is a gap between the last two expressions $s_n|_{\tau=2/\lambda-0} - s_n|_{\tau=2/\lambda+0} \sim \lambda^2/(q_n v)$, which is numerically small for most cases. The motion of the decay rates on the complex plane is schematically shown in Fig. 1. In the limit of strong disorder some of the s_n are on the real axis, and the imaginary part of s_n becomes nonzero when $\tau q_n v=1$. Then s_n moves along the arc and stops when $\tau=\lambda/2$.

Equations (10) and (16) show that a chaotic system is qualitatively different from a diffusive system from the point of view of the position of Ruelle's resonances s_n on the complex plane. In the chaotic limit all resonances lie on a line parallel to the imaginary axis. The disorder induces motion of the resonances toward the real axis as was found by Agam and Fishman [2].

The interpolation formula between Ruelle's type and Selberg's type zeta exists only if the diffusion modes transform to the so-called Frobenius-Perron modes as the disorder goes to zero. This has not yet been proven for our case. The main difficulty is that the diffusion modes are selected from all kinetic modes by the condition $\overline{f}\not\equiv 0$. At the same time Frobenius-Perron modes are selected by the choice of the functional space. However, in other systems, one can consider the diffusion modes as modes of the Frobenius-Perron operator [18].

Some additional information might be obtained from the properties of the propagator of Eq. (1), which can be written as a path integral for model 2,

$$G(\vec{r},\phi,\vec{r}_0,\phi_0,t)=\int D[\psi]\,\delta\!\left(\vec{r}-\vec{r}_0-\int_0^t \vec{v}_\psi dt\right) \times e^{-\tau/4\{\int_0^{t_1}\dot{\psi}^2dt+\int_{t_1}^{t_2}\dot{\psi}^2dt+\cdots+\int_{t_n}^{t}\dot{\psi}^2dt\}}, \tag{17}$$

where $\psi(0)=\phi_0, \psi(t_j-0)+\psi(t_j+0)=2\alpha_j, \ldots, \psi(t)=\phi$, and the path $\vec{r}_0+\int_0^t\vec{v}_\psi dt$ touches the boundaries n times at the points $\vec{r}_1,\ldots,\vec{r}_n$, at the times $t_1,\ldots,t_n$. The angle α_j is the direction of the tangent to the boundary at the reflection point $\vec{r}_j$. The trace of the propagator Eq. (17) known also as the return probability is

$$p(t)=\int d\vec{r}_0\int d\phi\int D[\psi]e^{-\tau/4\int_0^t\dot{\psi}^2dt}\,\delta\!\left(\int_0^t\vec{v}_\psi dt\right), \tag{18}$$

where $\psi(t)=\psi(0)=\phi$ and $\int_0^t\dot{\psi}^2dt$ is defined as in Eq. (17).

The propagator Eq. (17) should interpolate between the Frobenius-Perron operator in the limit $\tau\to\infty$ and the diffusion operator in the limit $\tau\to 0$. Then the trace of this propagator Eq. (18) should provide us with a systematic way to compute the interpolation formula for the zeta function, because $-\ln Z(s)=\int_0^\infty e^{st}t^{-1}p(t)dt$. Here the sign of st in the Laplace transform is positive, because we want the roots of the zeta function to have the meaning of the decay rates.

In the limit of weak disorder $\tau\to\infty$, one may hope to obtain the small corrections $\propto 1/\tau$ to the Frobenius-Perron operator, and, therefore, to Eq. (2). Particularly, one may expect to obtain the additional ''stabilization'' of the periodic orbits through the disorder. Let us consider the vicinity of the periodic orbit p in phase space. The path in such a vicinity can be described by the coordinate $x(t)=vt$ along the orbit, by the coordinate $y(t)$ normal to the orbit, and by the deviation of the direction of motion $\phi(t)$. The position of the particle at the end of the path and at the beginning of the path are connected by

$$\begin{pmatrix} y(t)\\ \phi(t)\end{pmatrix}=M_p\begin{pmatrix} y(0)\\ \phi(0)\end{pmatrix}+\sum_{j=1}^{n_p}M_{pj}\begin{pmatrix}\theta_j\frac{L_{pj}}{2}\\ \theta_j\end{pmatrix}, \tag{19}$$

where the orbit p crosses the billiard n_p times. In other words, the orbit consists of n_p segments of length L_{pj}. When the particle is going along the segment j, it can be scattered by the disorder at small angle θ_j, and then the rest of the path is distorted too. The cumulative change of the end of the path is given by the sum in the right-hand side of Eq. (19), where M_{pj} is the monodromy matrix of the piece of the orbit consisting of the segments $L_{pj+1},\ldots,L_{pn_p}$. One can see immediately from Eq. (19) that the stability amplitude of the closed path $y(t)=y(0),\phi(t)=\phi(0)$ is independent of $\theta_1,\ldots,\theta_{n_p}$, and, therefore, it is independent of τ. Therefore, there are no $1/\tau$ corrections to the zeta function Eq. (2) and our interpolation formula Eq. (15) is independent of τ for $\tau>2/\lambda$.

In the case of the three-dimensional billiards, the effect of the disorder is different because the scattering becomes three-dimensional and the distribution function depends on the three coordinates and two angles:

$$\frac{\partial f}{\partial t}+\vec{v}_{\theta\phi}\cdot\vec{\nabla}f=\frac{1}{\tau}\begin{cases}\overline{f}-f, & \text{model 1}\\ \nabla^2_{\theta\phi}f, & \text{model 2,}\end{cases} \tag{20}$$

where $\vec{v}_{\theta\phi}=(\sin\theta\cos\phi,\ \ \sin\theta\sin\phi,\ \ \cos\theta)$, $\nabla^2_{\theta\phi}\equiv(1/\sin\theta)(\partial/\partial\theta)\sin\theta(\partial/\partial\theta)+(1/\sin^2\theta)(\partial^2/\partial\phi^2)$ is the angular part of the Laplace operator, $\vec{f}\neq 0$, and the bar means the average over the solid angle. For the cubic billiard the spatial dependence of the density is again $\Sigma e^{i\vec{q}\cdot\vec{r}}$, where the sum is over the six orthogonal orientations of $\vec{q}$, and the modes are selected by the rule $\sin(q_xL)\sin(q_yL)\sin(q_zL)=0$. Then, the dispersion relation for the model 1, (uniform scattering), can be found in Ref. [19], Eq. (12.2.11):

$$1-s\tau=\frac{qv\tau}{\tan(qv\tau)}, \tag{21}$$

where $qv\tau<\pi$. In other words there are no modes with $qv\tau\geqslant\pi$. Equation (21) describes the diffusion modes for small τ, but it cannot be used for large τ. If the mode has q close to $\pi/(v\tau)$, then the decay of such a mode is very fast $s\propto(\pi-qv\tau)^{-1}$.

The model with small angle scattering in three dimensions is the Fokker-Planck equation for the distribution function, which should be solved together with mirror boundary conditions on the billiard boundary. The solutions inside the cubic billiard have the dispersion $s(q)$ similar to that of the square billiard, if $\overline{f}\neq 0$. Therefore, one may hope that Eq. (15) gives the interpolation of the zeta function of the Fokker-Planck equation for modes with $\overline{f}\neq 0$.

In summary, we have constructed the interpolation formula for the zeta function of the kinetic equation, in both ''chaotic,'' Eqs. (15), and ''integrable,'' Eq. (13), cases. Our zeta function describes the modes with nonzero angle average only, i.e., modes with nonzero density of particles. From the mathematical point of view, our kinetic zeta function interpolates between Ruelle's and Selberg's zeta functions. Our formulas are independent of the particular choice of the collision integral for two-dimensional billiards and are suitable for small angle scattering in three dimensions.

It is my pleasure to thank Professor Smilansky for important remarks and discussions. This work was supported by the Israel Science Foundation and the Minerva Center for Nonlinear Physics of Complex Systems.

[1] A. Atland and Y. Gefen, Phys. Rev. B **51**, 10 671 (1995).
[2] O. Agam and S. Fishman, J. Phys. A **29**, 2013 (1996).
[3] I. L. Aleiner and A. I. Larkin, Phys. Rev. B **54**, 14 423 (1996).
[4] C. M. Marcus, S. R. Patel, A. G. Huibers, S. M. Cronenwett, M. Switkes, I. H. Chan, R. M. Clarke, J. A. Folk, S. F. Godijn, K. Campman, and A. C. Gossard, Chaos Solitons Fractals **8**, 1261 (1997).
[5] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford, 1981), Chap. I, Sec. 19.
[6] L. A. Bunimovich, Zh. Éksp. Teor. Fiz. **89**, 1452 (1985) [Sov. Phys. JETP **62**, 842 (1985)].
[7] P. Cvitanovic and B. Eckardt, J. Phys. A **24**, 237 (1991).
[8] M. G. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990),
[9] A. Voros, J. Phys. A **21**, 685 (1988).
[10] M. Sieber, H. Primack, U. Smilansky, I. Ussishkin, and H. Schanz, J. Phys. A **28**, 5041 (1995).
[11] P. Gaspard, in *Quantum Chaos*, Proceedings of the International School of Physics ''Enrico Fermi,'' Course CXIX, Varenna, 1991, edited by G. Casati, I. Guarneri, and U. Smilansky (North-Holland, Amsterdam, 1993), p. 307.
[12] P. Gaspard and S. A. Rice, J. Chem. Phys. **90**, 2255 (1989).
[13] P. Gaspard and D. A. Ramirez, Phys. Rev. A **45**, 8383 (1992).
[14] P. Cvitanovic, P. E. Rosenqvist, G. Vattay, and H. H. Rugh, Chaos **3**, 619 (1993).
[15] M. V. Berry, Proc. R. Soc. London, Ser. A **400**, 229 (1985).
[16] M. V. Berry and M. Tabor, J. Phys. A **10**, 371 (1977).
[17] U. Smilansky, in *Mesoscopic Quantum Physics*, edited by J.-L. P. E. Akkermans, G. Motambaux, and J. Zinn-Justin (Elsevier, Amsterdam, 1995), Chap. 7, p. 373.
[18] P. Gaspard, Phys. Rev. E **53**, 4379 (1996).
[19] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 12.